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GLOBAL EXISTENCE FOR THE DEFOCUSING NONLINEAR SCHRÖDINGER EQUATIONS WITH LIMIT PERIODIC INITIAL DATA

TADAHIRO OH

Dedicated to Professor Gustavo Ponce on the occasion of his sixtieth birthday

ABSTRACT. We consider the Cauchy problem for the defocusing nonlinear Schrödinger equations (NLS) on the real line with a special subclass of almost periodic functions as initial data. In particular, we prove global existence of solutions to NLS with limit periodic functions as initial data under some regularity assumption.

1. INTRODUCTION

We consider the Cauchy problem for the defocusing nonlinear Schrödinger equations (NLS) on \mathbb{R} :

$$\begin{cases} i\partial_t u + \partial_x^2 u = |u|^{2k}u, \\ u|_{t=0} = f, \end{cases} \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.1)$$

for $k \in \mathbb{N}$. The Cauchy problem (1.1) has been studied extensively in terms of the usual Sobolev spaces $H^s(\mathbb{R})$ on the real line and the Sobolev spaces $H_{\text{per}}^s(\mathbb{R}) \simeq H^s(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, of periodic functions (of a fixed period) on \mathbb{R} . See, for example, Ginibre-Velo [12], Tsutsumi [19], and Bourgain [3]. There are several known conservation laws for NLS (1.1). In particular, the conservation of the Hamiltonian and the mass, defined by

$$\begin{aligned} \text{Hamiltonian: } H[u](t) &= \frac{1}{2} \int_{\mathcal{M}} |\partial_x u(t, x)|^2 dx + \frac{1}{2k+2} \int_{\mathcal{M}} |u(t, x)|^{2k+2} dx, \\ \text{Mass: } Q[u](t) &= \int_{\mathcal{M}} |u(t, x)|^2 dx, \end{aligned}$$

where $\mathcal{M} = \mathbb{R}$ or \mathbb{T} , plays a crucial role in establishing global well-posedness of (1.1). See also the monographs [7, 17] for more references on the subject.

Our main interest in this paper is to study global-in-time behavior of solutions to the Cauchy problem (1.1) with a particular subclass of almost periodic functions as initial data. In particular, we prove global existence

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of unique solutions to (1.1) with *limit periodic* functions (see Definitions 1.6 and 1.8 below) as initial data under some regularity assumption.

In Subsection 1.1, we go over the basic definitions and properties of almost periodic functions along with the known well-posedness results for NLS (1.1) with almost periodic functions as initial data. We then introduce limit periodic functions and state our main result (Theorem 1.9) in Subsection 1.2.

1.1. Almost periodic functions. Let us first recall the definition of almost periodic functions due to Bohr [2].

Definition 1.1. We say that a function f on \mathbb{R} is *almost periodic*, if it is continuous and, for every $\varepsilon > 0$, there exists $L = L(\varepsilon, f) > 0$ such that every interval of length L on \mathbb{R} contains a number τ such that

$$\sup_{x \in \mathbb{R}} |f(x - \tau) - f(x)| < \varepsilon.$$

We use $AP(\mathbb{R})$ to denote the space of almost periodic functions on \mathbb{R} .

In the following, we briefly go over some basic properties of almost periodic functions. See Besicovitch [1], Corduneanu [9], and Katznelson [13] for more on the subject. It is well known that the following two notions in Definition 1.2 (i) and (ii) are equivalent to the notion of almost periodic functions in Definition 1.1.

Definition 1.2. (i) We say that a function f on \mathbb{R} has the *approximation property*, if it can be uniformly approximated by trigonometric polynomials (of finite degrees).

(ii) We say that a continuous function on \mathbb{R} is *normal* if, given any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}$, the collection $\{f(\cdot + x_n)\}_{n=1}^{\infty}$ is precompact in $L^{\infty}(\mathbb{R})$.

Remark 1.3. Definition 1.2 (i) states that the collection $AP(\mathbb{R})$ of almost periodic functions is exactly the closure of the trigonometric polynomials with respect to the uniform metric induced by the L^{∞} -norm. There are also the notions of different classes of generalized almost periodic functions due to Stepanov, Weyl, and Besicovitch by considering the closures of the trigonometric polynomials under different metrics. The corresponding spaces are denoted by S^p , W^p , and B^p , respectively. Note that we have $AP(\mathbb{R}) \subset S^p \subset W^p \subset B^p$, $p \geq 1$. For example, see Remarks 1.4 and 1.14 below. In the literature, almost periodic functions in $AP(\mathbb{R})$ are sometimes referred to as *uniformly* almost periodic functions in order to distinguish them from these generalized almost periodic functions. In this paper, however, we only consider almost periodic functions in Bohr's sense according to Definition 1.1.

The space $AP(\mathbb{R})$ of almost periodic functions is a closed subalgebra of $L^\infty(\mathbb{R})$ and almost periodic functions are uniformly continuous. Given $f \in AP(\mathbb{R})$, we can define the so-called mean value $M(f)$ of f by

$$M(f) := \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L f(x) dx.$$

We then define an inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$ on $AP(\mathbb{R})$ by

$$\langle f, g \rangle_{\mathcal{L}^2} := M(f\bar{g}) = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L f(x)\bar{g}(x) dx$$

for $f, g \in AP(\mathbb{R})$. Note that $M(f)$ and $\langle f, g \rangle_{\mathcal{L}^2}$ are well defined for $f, g \in AP(\mathbb{R})$. This inner product induces the \mathcal{L}^2 -norm defined by

$$\|f\|_{\mathcal{L}^2} := M(|f|^2) = \langle f, f \rangle_{\mathcal{L}^2} \quad (1.2)$$

and makes the space $AP(\mathbb{R})$ of almost periodic functions a pre-Hilbert space (missing completeness). Note that the \mathcal{L}^2 -norm is a norm on $AP(\mathbb{R})$, but not for general functions on \mathbb{R} . For example, we have $\|f\|_{\mathcal{L}^2} = 0$ for any bounded function f with a compact support.

Remark 1.4. The space B^p , $p \geq 1$, of Besicovitch's generalized almost periodic functions is precisely the closure of the trigonometric polynomials under the B^p -metric d_{B^p} defined by

$$d_{B^p}(f, g) := \limsup_{L \rightarrow \infty} \left(\frac{1}{2L} \int_{-L}^L |f(x) - g(x)|^p dx \right)^{\frac{1}{p}}.$$

It is known that B^2 is complete with respect to the \mathcal{L}^2 -norm defined in (1.2). See [1, Riesz-Fischer Theorem on p. 109].

Now, let us turn our attention to the Fourier analysis of almost periodic functions. The complex exponentials $\{e^{2\pi i \omega x}\}_{\omega \in \mathbb{R}}$ form an orthonormal family under the inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}^2}$. Given $f \in AP(\mathbb{R})$, we then define its Fourier coefficient by

$$\hat{f}(\omega) := \langle f, e^{2\pi i \omega x} \rangle_{\mathcal{L}^2} = M(fe^{-2\pi i \omega x}). \quad (1.3)$$

It follows from Bessel's inequality that $\hat{f}(\omega) = 0$ except for countable many values of $\omega \in \mathbb{R}$. Given $f \in AP(\mathbb{R})$, we define the Fourier series associated to f by

$$f(x) \sim \sum_{\omega \in \sigma(f)} \hat{f}(\omega) e^{2\pi i \omega x}. \quad (1.4)$$

Here, $\sigma(f)$ denotes the frequency set of f defined by

$$\sigma(f) := \{\omega \in \mathbb{R} : \hat{f}(\omega) \neq 0\}.$$

It is known that the orthonormal family $\{e^{2\pi i\omega x}\}_{\omega \in \mathbb{R}}$ is complete in the sense that two distinct almost periodic functions have distinct Fourier series. Moreover, we have the Parseval's identity:

$$\|f\|_{\mathcal{L}^2} = \left(\sum_{\omega \in \mathbb{R}} |\widehat{f}(\omega)|^2 \right)^{\frac{1}{2}}$$

for $f \in AP(\mathbb{R})$. Regarding the actual convergence of the Fourier series to an almost periodic function, we have the following lemma.

Lemma 1.5 (Theorem 1.20 in [9]). *Let $f \in AP(\mathbb{R})$. If the Fourier series associated to f converges uniformly on \mathbb{R} , then it converges to f . Namely, we have*

$$f(x) = \sum_{\omega \in \sigma(f)} \widehat{f}(\omega) e^{2\pi i\omega x}. \quad (1.5)$$

Since our argument in this paper is based on the Fourier series representation (1.5), Lemma 1.5 plays an important role in the following.

Given $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^\infty \in \mathbb{R}^\mathbb{N}$, we say that $\boldsymbol{\omega}$ is linear independent if any relation of the form:

$$\sum_{j=1}^N \alpha_j \omega_j = 0, \quad \alpha_j \in \mathbb{Q},$$

implies that $\alpha_j = 0$, $j = 1, \dots, N$.

Given a set S of real numbers, we say that a linearly independent set $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^\infty$ is a basis for the set S , if every element in S can be represented as a finite linear combination of elements in $\boldsymbol{\omega}$ with rational coefficients. Given $f \in AP(\mathbb{R})$, we say that a linearly independent set $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N$, allowing the case $N = \infty$, is a basis of f , if it is a basis for the frequency set $\sigma(f)$ of f . Lemma 1.14 in [9] guarantees existence of a basis of $f \in AP(\mathbb{R})$. We say that a basis $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N$ of f is an integral basis if any element in the frequency set $\sigma(f)$ can be written as a finite linear combination of elements in $\boldsymbol{\omega}$ with integer coefficients. If there exists a finite integral basis of f , i.e. $N < \infty$, then we say that the function f is *quasi-periodic*.

We conclude this subsection by going over the known well-posedness results on NLS with almost periodic functions as initial data. In the periodic setting, Bourgain [3] proved local well-posedness of (1.1) in $H^s(\mathbb{T})$ with $s = 0$ when $k = 1$ and with $s > \frac{1}{2} - \frac{1}{k}$ when $k \geq 2$. The conservation of the Hamiltonian and the mass then yields global well-posedness of (1.1) in $L^2(\mathbb{T})$ when $k = 1$ and in $H^1(\mathbb{T})$ when $k \geq 2$.

Regarding local well-posedness with quasi-periodic initial data, Tsugawa [18] considered quasi-periodic functions f of the form:

$$f(x) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n}) e^{2\pi i(\boldsymbol{\omega} \cdot \mathbf{n})x}, \quad (1.6)$$

for a frequency set $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^N \in \mathbb{R}^N$ with $N \in \mathbb{N}$. Then, he proved local well-posedness of the Korteweg-de Vries equation (KdV) on \mathbb{R} :

$$\partial_t u + \partial_x^3 u = u \partial_x u$$

with quasi-periodic initial data of the form (1.6) under some regularity condition. Moreover, by defining in a Sobolev-type space $\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})$ of quasi-periodic functions of the form (1.6) by the norm

$$\|f\|_{\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})} := \|\langle \mathbf{n} \rangle^{\mathbf{s}} \widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n})\|_{\ell_{\mathbf{n}}^2(\mathbb{Z}^N)}, \quad \langle \mathbf{n} \rangle^{\mathbf{s}} := \prod_{j=1}^N (1 + |n_j|^2)^{\frac{s_j}{2}},$$

where $\mathbf{s} = \{s_j\}_{j=1}^N \in \mathbb{R}^N$, [18, Lemma 2.2 (i)] implies that NLS (1.1) is locally well-posed in $\mathcal{H}_{\boldsymbol{\omega}}^{\mathbf{s}}(\mathbb{R})$, provided that $\min(s_1, \dots, s_N) > \frac{1}{2}$.

Let us turn our attention to the generic almost periodic setting. Fix a frequency set $\boldsymbol{\omega} = \{\omega_j\}_{j=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. For almost periodic functions $f \in AP(\mathbb{R})$ of the form:

$$f(x) \sim \sum_{\mathbf{n} \in \mathbb{Z}^{\mathbb{N}}} \widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n}) e^{2\pi i(\boldsymbol{\omega} \cdot \mathbf{n})x}, \quad (1.7)$$

where $\mathbf{n} = \{n_j\}_{j=1}^{\infty} \in \mathbb{Z}^{\mathbb{N}}$, we define the $\mathcal{A}_{\boldsymbol{\omega}}$ -norm by

$$\|f\|_{\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})} = \|\widehat{f}(\boldsymbol{\omega} \cdot \mathbf{n})\|_{\ell_{\mathbf{n}}^1(\mathbb{Z}^{\mathbb{N}})}.$$

Then, we define the algebra $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ by

$$\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R}) = \{f \in AP(\mathbb{R}) : \begin{array}{l} f \text{ is of the form (1.7) and } \|f\|_{\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})} < \infty \end{array}\}. \quad (1.8)$$

See [16] for some basic properties of $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$. In [16], we proved local well-posedness in $\mathcal{A}_{\boldsymbol{\omega}}(\mathbb{R})$ for NLS with a power-type nonlinearity, including (1.1).

In view of the result in [16], it is natural to consider the global-in-time behavior of solutions to (1.1). This is an extremely difficult question in general. There are, however, several known global existence results for the cubic NLS, (1.1) with $k = 1$, and KdV in the almost periodic and quasi-periodic settings. Egorova [11] and Boutet de Monvel-Egorova [5] constructed global-in-time solutions to KdV and the cubic NLS with almost periodic initial data, assuming some conditions, including Cantor-like spectra for the corresponding Schrödinger operator (for KdV) and Dirac operator (for the cubic NLS). In particular, the class of almost periodic initial data in [11, 5] includes almost periodic functions f that can be approximated by periodic functions f_j of growing periods $\alpha_j \rightarrow \infty$ in $S^{s,2}(\mathbb{R})$ with $s \geq 4$ for KdV and $s \geq 3$ for cubic NLS. See (1.13) below for the definition of the $S^{s,2}$ -norm. Moreover, the convergence of f_j to f in the $S^{s,2}$ -norm is assumed to be exponentially fast. It is worthwhile to mention that the solutions constructed in [11, 5] are almost periodic in both t and x . There is also a recent global existence

result of KdV with quasi-periodic initial data by Damanik-Goldstein [10]. Their result states that if the Fourier coefficient $\widehat{f}(\boldsymbol{\omega} \cdot \boldsymbol{n})$ of a quasi-periodic initial condition (1.6) decays exponentially fast (in \boldsymbol{n}), then there exists a unique global solution whose Fourier coefficient also decays exponentially fast (with a slightly worse constant), provided that a smallness condition on the initial condition f and a Diophantine condition on $\boldsymbol{\omega}$ are satisfied.

We emphasize that the above results rely heavily on the inverse spectral method and on the complete integrability of the equations. These methods are not applicable to the non-integrable case, i.e. (1.1) with $k \geq 2$. Our main goal in this paper is to establish global existence via an analytical method without complete integrability.

1.2. Limit periodic functions and the main result. In the following, we restrict our attention to a particular subclass of almost periodic functions, called limit periodic functions.

Definition 1.6. We say that a function f on \mathbb{R} is *limit periodic* if it is a uniform limit of continuous periodic functions.

Note that a limit periodic function is almost periodic, since $AP(\mathbb{R})$ is closed under the L^∞ -norm. The following characterization of limit period functions plays an essential role in our analysis.

Lemma 1.7 (Theorem and Converse Theorem on p.32 in [1]). *An almost periodic function f on \mathbb{R} is limit periodic if and only if its Fourier series is given by*

$$f(x) \sim \sum_{m=1}^{\infty} \widehat{f}(r_m \omega) e^{2\pi i r_m \omega x} \quad (1.9)$$

for some $\omega \in \mathbb{R}$ and $\{r_m\}_{m=1}^{\infty} \subset \mathbb{Q}$.

Namely, an almost periodic function is limit periodic if and only if it has a one-term basis, i.e. all the frequencies are rational multiples of a single frequency $\omega \in \mathbb{R}$.

In view of the global well-posedness result [3] in the periodic setting, we assume that our initial condition f is not periodic. Note that an almost periodic function is periodic if and only if it has a one-term *integral* basis. Hence, we assume that ω (or any of its rational multiple) is not an integral basis.¹ It also follows from Lemma 1.7 that if a limit periodic function is quasi-periodic, then it is periodic. Therefore, we consider limit periodic functions that are not quasi-periodic in the following. Lastly, we also assume that $\omega \neq 0$ in the following, since $\omega = 0$ corresponds to constant functions.

¹This in particular implies that the denominators of $\{r_m\}_{m \in \mathbb{N}}$ are unbounded.

Definition 1.8. Let $\omega \in \mathbb{R} \setminus \{0\}$. We denote the class of limit periodic functions with a one-term basis $\omega \in \mathbb{R}$ by $LP_\omega(\mathbb{R})$. Namely, we say that $f \in LP_\omega(\mathbb{R})$ if it has the Fourier series expansion (1.9) with this specific ω .

Given a limit periodic function f , define f_j , $j \in \mathbb{N}$, by

$$f_j(x) \sim \sum_{m \in A(j)} \widehat{f}(r_m \omega) e^{2\pi i(r_m j!) \frac{\omega}{j!} x}, \quad (1.10)$$

where $A(j)$ is given by

$$A(j) = \{m \in \mathbb{N} : r_m j! \in \mathbb{Z}\}. \quad (1.11)$$

Then, f_j is periodic with period

$$L_j := \frac{j!}{\omega}. \quad (1.12)$$

Moreover, it is known that f_j converges to f uniformly as $j \rightarrow \infty$. See [1, p.45]. We refer to f_j as the periodization of f (with period L_j). See Lemma 1.15 below.

Let $p \geq 1$ and $s \in \mathbb{N}$. We define our function space $S^{s,p}(\mathbb{R})$ by

$$S^{s,p}(\mathbb{R}) = \{f \in L^1_{\text{loc}}(\mathbb{R}) : \|f\|_{S^{s,p}(\mathbb{R})} < \infty\},$$

where the $S^{s,p}$ -norm is defined by

$$\|f\|_{S^{s,p}(\mathbb{R})} := \sup_{y \in \mathbb{R}} \left(\int_y^{y+1} |f(x)|^p + |\partial_x^s f(x)|^p dx \right)^{\frac{1}{p}}. \quad (1.13)$$

Note that $S^{s,p}(\mathbb{R})$ is complete,² just like the usual Sobolev spaces $W^{s,p}([y, y+1])$. On the one hand, the definition of $S^{s,p}(\mathbb{R})$ has nothing to do with almost periodic functions. On the other hand, we point out that the $S^{s,p}$ -norm is closely related to the S^p -metric used for Stepanov's generalized almost periodic functions. See Remark 1.14 below.

Now, we are ready to state our main result.

Theorem 1.9. *Given $\omega \in \mathbb{R} \setminus \{0\}$, let $f \in LP_\omega(\mathbb{R}) \cap S^{1,2}(\mathbb{R})$ and f_j be the periodization of f as in (1.10).*

(i) *Suppose that there exist $\varepsilon > 0$ and $B > 0$ such that*

$$e^{L_{j+1}^{2k+\varepsilon}} \|f_j - f\|_{S^{1,2}(\mathbb{R})} \leq B \quad (1.14)$$

²To see this, we can consider the following $\widetilde{S}^{s,p}$ -norm given by

$$\|f\|_{\widetilde{S}^{s,p}(\mathbb{R})} := \sup_{j \in \mathbb{Z}} \left(\int_j^{j+1} |f(x)|^p + |\partial_x^s f(x)|^p dx \right)^{\frac{1}{p}}.$$

Clearly, the $S^{s,p}$ - and $\widetilde{S}^{s,p}$ -norms are equivalent. Moreover, $\widetilde{S}^{s,p}(\mathbb{R})$ is complete, since $W^{s,p}([j, j+1])$ is complete for each $j \in \mathbb{Z}$.

for all sufficiently large j . Then, there exists a unique global solution $u \in C(\mathbb{R}; S^{1,2}(\mathbb{R})) \subset C(\mathbb{R}; L^\infty(\mathbb{R}))$ to the defocusing NLS (1.1) with $u|_{t=0} = f$. Moreover, $u(t)$ lies in $LP_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$.

(ii) Given $J \in \mathbb{N}$ and $K > 0$, define $\mathcal{B}^\omega(J, K)$ by

$$\mathcal{B}^\omega(J, K) = \{f \in LP_\omega(\mathbb{R}) \cap S^{1,2}(\mathbb{R}) : \|f\|_{S^{1,2}(\mathbb{R})} \leq K, \quad (1.14) \text{ is satisfied for all } j \geq J\}. \quad (1.15)$$

Then, for fixed $t \in \mathbb{R}$, the solution map: $u(0) = f \mapsto u(t)$ constructed in (i) is continuous on $\mathcal{B}^\omega(J, K)$ with the $S^{1,2}$ -topology.

To the best of the author's knowledge, Theorem 1.9 is the first global existence result for NLS (1.1), $k \geq 2$, with limit periodic functions as initial data. While global existence of solutions for the cubic NLS ($k = 1$) was previously proved in [5], the argument in [5] relied heavily on the complete integrability of the cubic NLS and is not applicable to the non-integrable case $k \geq 2$. We prove Theorem 1.9 by combining global well-posedness of the defocusing NLS in the periodic setting and scaling invariance.

Remark 1.10. (i) Given $f \in LP_\omega(\mathbb{R})$, let r_m be as in (1.9). Then, under the hypothesis of Theorem 1.9, we prove that $u(t) \in A_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$, where

$$\omega := \{r_m \omega\}_{m=1}^\infty \in \mathbb{R}^\mathbb{N}.$$

Then, from the local well-posedness in $\mathcal{A}_\omega(\mathbb{R})$ [16], we obtain uniqueness and (local-in-time) continuous dependence, at each $t \in \mathbb{R}$, in the $\mathcal{A}_\omega(\mathbb{R})$ -topology of the global-in-time flow constructed in Theorem 1.9.

(ii) The uniqueness statement in Theorem 1.9 holds in $C(\mathbb{R}; \mathcal{A}_\omega(\mathbb{R}))$ as mentioned above. There is also a mild uniqueness statement³ as a limit of the periodic solutions u_j to (1.1) with $u_j|_{t=0} = f_j$, where f_j is as in (1.10). At this point, however, we do not know how to prove uniqueness in $C(\mathbb{R}; S^{1,2}(\mathbb{R}))$.

(iii) It would be of interest to characterize limit periodic functions satisfying (1.14). In Appendix A, we present a brief discussion on a sufficient condition for (1.14).

Remark 1.11. Let us compare Theorem 1.9 and the result in [5], when $k = 1$. On the one hand, the rate of approximation of f by the periodization f_j (i.e. the condition (1.14) in Theorem 1.9) is more restrictive than that in [5]. On the other hand, Theorem 1.9 holds with $s = 1$, while the result in [5] requires a higher regularity $s \geq 3$.

In Theorem 1.9, we set $s = 1$. Indeed, it is possible to lower the value of s in view of global well-posedness of the defocusing periodic NLS (in

³This is analogous to having uniqueness only as a limit of classical solutions to some evolution equations (even in the usual Sobolev spaces).

particular, for $k = 1$ and 2 ; see [3, 4]). In this case, one needs to (i) control the growth of the H^s -norm of a solution to the periodic problem in the spirit of the I -method [8] and (ii) adjust the convergence rate (1.14) to the regularity $s < 1$. In our almost/limit periodic setting, it is important to control the L^∞ -norm (in x). Hence, we need $s > \frac{1}{2}$ in view of Sobolev embedding theorem.

Remark 1.12. While we state and prove Theorem 1.9 only for the defocusing case, the global existence result also holds in the focusing case when $k = 1$. In this case, we need to replace (1.14) by

$$e^{L_{j+1}^{6+\varepsilon}} \|f_j - f\|_{S^{1,2}(\mathbb{R})} \leq B \quad (1.16)$$

for all sufficiently large j . See Remark 3.1.

In the focusing case with $k \geq 2$, finite time blowup solutions are known to exist in $H^1(\mathbb{R})$ and $H^1(\mathbb{T})$. In these settings, there are some criteria on such finite time blowup solutions (such as negative energy [14, 15]). Since a periodic function is in particular almost periodic, finite time blowup solutions in the periodic setting [14] provide an instance of finite time blowup results in the almost periodic setting (where initial data are periodic). It would be interesting to provide a criterion for finite time blowup solutions to (1.1) in a generic (i.e. non-periodic) almost periodic setting.

In [16], we studied the following NLS:

$$i\partial_t u + \partial_x^2 u = \lambda |u|^{2k} \quad (1.17)$$

in a generic almost periodic setting and provided a criterion for finite time blowup solutions, depending only on the signs of the real and imaginary parts of λ and the mean value $M(f)$ of an almost periodic initial condition f .

Remark 1.13. A version of Morrey's inequality states that

$$|f(x) - f(x_0)| \leq Cr^{1-\frac{d}{p}} \left(\int_{B(x_0, 2r)} |\nabla f(y)|^p dy \right)^{\frac{1}{p}} \quad (1.18)$$

for all $x \in B(x_0, r)$ and $d < p \leq \infty$. Here, $B(x_0, r) \subset \mathbb{R}^d$ denotes the ball of radius r centered at $x_0 \in \mathbb{R}^d$. Then, given a limit periodic function $f \in S^{1,2}(\mathbb{R})$, it follows from (1.18) that $f \in C^{\frac{1}{2}}(\mathbb{R})$. Hence, the Fourier series of f converges to f uniformly. See [1, p. 46]. In particular, by Lemma 1.5, we conclude that the function f is indeed represented by its Fourier series.

Remark 1.14. The space S^p , $p \geq 1$, of Stepanov's generalized almost periodic functions is precisely the closure of the trigonometric polynomials

under the S^p -metric d_{S^p} defined by

$$d_{S^p}(f, g) := \sup_{y \in \mathbb{R}} \left(\int_y^{y+1} |f(x) - g(x)|^p dx \right)^{\frac{1}{p}}. \quad (1.19)$$

Note that this metric is induced by the $S^{s,p}$ -norm with $s = 0$. Unlike (uniformly) almost periodic functions, the S^p -generalized almost periodic functions are determined up to sets of measure 0. On the one hand, an almost periodic function is uniformly continuous. On the other hand, if $f \in S^p$ is uniformly continuous on \mathbb{R} , then it is (uniformly) almost periodic. See [9, Theorem 6.16 on p. 174].

We conclude this introduction by stating a useful lemma, allowing us to extract a periodic component from an almost periodic function. Let f be a function on \mathbb{R} . Given $n \in \mathbb{N}$ and $L > 0$, define the averaging operator $A_{n,L}$ by

$$A_{n,L}[f](x) := \frac{1}{n} \left\{ f(x) + f(x+L) + \cdots + f(x+(n-1)L) \right\}. \quad (1.20)$$

Then, we have the following convergence property of the averaging operator on almost periodic functions.

Lemma 1.15 (Theorem on p. 44 of [1]). *Let f be an almost periodic function with the Fourier series (1.4). Then, for each $L > 0$, the limit*

$$f^{(L)}(x) := \lim_{n \rightarrow \infty} A_{n,L}[f](x) \quad (1.21)$$

exists uniformly in $x \in \mathbb{R}$. Moreover, $f^{(L)}$ is a periodic function with period L whose Fourier series consists of the terms of the Fourier series (1.4) of f which have period L . Namely,

$$f^{(L)}(x) \sim \sum_{\omega \in \mathbb{Z}/L} \hat{f}(\omega) e^{2\pi i \omega x}.$$

In the following, we refer to $f^{(L)}$ as the periodization of f (with period L).

2. SOBOLEV SPACES ON A SCALED TORUS AND SCALING INVARIANCE OF NLS

In this section, we briefly go over the basic definitions and properties of Sobolev spaces on a scaled torus $\mathbb{T}_\lambda := \mathbb{R}/(\lambda\mathbb{Z})$, $\lambda \geq 1$, along with the scaling symmetry of NLS (1.1). Given a function F on \mathbb{T}_λ , we define its Fourier coefficient by

$$\hat{F}\left(\frac{n}{\lambda}\right) = \frac{1}{\lambda} \int_{\mathbb{T}_\lambda} F(x) e^{-2\pi i \frac{n}{\lambda} x} dx, \quad n \in \mathbb{Z}. \quad (2.1)$$

We have the following Fourier inversion formula:

$$F(x) = \sum_{n \in \mathbb{Z}} \hat{F}\left(\frac{n}{\lambda}\right) e^{2\pi i \frac{n}{\lambda} x} \quad (2.2)$$

and Plancherel's identity:

$$\|F\|_{L^2(\mathbb{T}_\lambda)} = \lambda^{\frac{1}{2}} \|\widehat{F}\|_{\ell^2(\mathbb{Z}/\lambda)} = \lambda^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} |\widehat{F}(\frac{n}{\lambda})|^2 \right)^{\frac{1}{2}}.$$

Note that the definition (2.1) of the Fourier coefficient for periodic functions agrees with the definition (1.3) of the Fourier coefficient for almost periodic functions.

Next, we define the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{T}_\lambda)$ and the inhomogeneous Sobolev spaces $H^s(\mathbb{T}_\lambda)$ by the norms:

$$\|F\|_{\dot{H}^s(\mathbb{T}_\lambda)} := \lambda^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} |\frac{n}{\lambda}|^{2s} |\widehat{F}(\frac{n}{\lambda})|^2 \right)^{\frac{1}{2}}, \quad (2.3)$$

$$\|F\|_{H^s(\mathbb{T}_\lambda)} := \lambda^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} \langle \frac{n}{\lambda} \rangle^{2s} |\widehat{F}(\frac{n}{\lambda})|^2 \right)^{\frac{1}{2}}, \quad (2.4)$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

NLS (1.1) on \mathbb{R} enjoys several symmetries. In particular, the scaling symmetry plays an essential role in the proof of Theorem 1.9. The scaling symmetry states that if u is a solution to (1.1) with initial condition ϕ , then the scaled function u^λ , defined by

$$u^\lambda(t, x) = \mathfrak{S}_\lambda[u](t, x) := \lambda^{-\frac{1}{k}} u(\lambda^{-2}t, \lambda^{-1}x), \quad (2.5)$$

is also a solution to (1.1) with the scaled initial condition:

$$f^\lambda(x) = \mathfrak{S}_\lambda[f](x) := \lambda^{-\frac{1}{k}} f(\lambda^{-1}x). \quad (2.6)$$

With a slight abuse of notation, we use \mathfrak{S}_λ to denote both the dilation operator (2.5) for functions in t and x and the dilation operator (2.6) for functions only in x , depending on the context.

We conclude this section by discussing the effect of the scaling (2.6) on different norms. Let f be a function on \mathbb{T} . It follows from (2.1) and (2.6) that the Fourier coefficient of the scaled function $f^\lambda := \mathfrak{S}_\lambda[f]$ on \mathbb{T}_λ is given by

$$\widehat{f^\lambda}(\frac{n}{\lambda}) = \lambda^{-\frac{1}{k}} \widehat{f}(n). \quad (2.7)$$

Then, from (2.3), we have

$$\|f^\lambda\|_{\dot{H}^s(\mathbb{T}_\lambda)} = \lambda^{\frac{1}{2}-s-\frac{1}{k}} \|f\|_{\dot{H}^s(\mathbb{T})}. \quad (2.8)$$

In particular, for $\lambda \geq 1$ and $s \geq 0$, we have

$$\|f\|_{H^s(\mathbb{T})} \lesssim \lambda^{-\frac{1}{2}+s+\frac{1}{k}} \|f^\lambda\|_{H^s(\mathbb{T}_\lambda)} \quad (2.9)$$

Let F be a function on \mathbb{T}_λ . Then, from (2.2) and (2.3), we have the following Sobolev embedding type estimate:

$$\|F\|_{L^\infty(\mathbb{T}_\lambda)} \leq |\widehat{F}(0)| + C\lambda^{s-\frac{1}{2}}\|F\|_{\dot{H}^s(\mathbb{T}_\lambda)}, \quad (2.10)$$

as long as $s > \frac{1}{2}$. Combining (2.7), (2.8), and (2.10), we obtain

$$\|f^\lambda\|_{L^\infty(\mathbb{T}_\lambda)} \leq \lambda^{-\frac{1}{k}}|\widehat{f}(0)| + C\lambda^{-\frac{1}{k}}\|f\|_{\dot{H}^s(\mathbb{T})} \sim \lambda^{-\frac{1}{k}}\|f\|_{H^s(\mathbb{T})} \quad (2.11)$$

for f on \mathbb{T} as long as $s > \frac{1}{2}$.

Lastly, we state a version of Sobolev embedding on \mathbb{T}_λ . By Cauchy-Schwarz inequality along with a Riemann sum approximation, we have

$$\begin{aligned} \|F\|_{L^\infty(\mathbb{T}_\lambda)} &\leq \|\widehat{F}\|_{\ell^1(\mathbb{Z}/\lambda)} \leq \left(\frac{1}{\lambda} \sum_{n \in \mathbb{Z}} \frac{1}{(1 + |\frac{n}{\lambda}|^2)^s} \right)^{\frac{1}{2}} \|F\|_{H^s(\mathbb{T}_\lambda)} \\ &\lesssim \|F\|_{H^s(\mathbb{T}_\lambda)} \end{aligned} \quad (2.12)$$

for $s > \frac{1}{2}$. Here, the implicit constants are independent of the period λ .

Remark 2.1. Note that (2.12) with (2.8) only yields

$$\|f^\lambda\|_{L^\infty(\mathbb{T}_\lambda)} \lesssim \lambda^{\frac{1}{2}-\frac{1}{k}}\|f\|_{H^s(\mathbb{T})}$$

for $s > \frac{1}{2}$, which is not as efficient as (2.11). This is due to the fact that the homogeneous Sobolev norms act better than the inhomogeneous Sobolev norms with respect to the scaling. In the following, we will use both (2.11) and (2.12).

3. GLOBAL EXISTENCE

In this section, we present the proof of Theorem 1.9. The proof is based on an elementary scaling argument and the H^1 -global well-posedness of the defocusing NLS in the periodic setting. We first introduce some notations. Given $j \in \mathbb{N}$, we set

$$\mathbb{T}_j := \mathbb{R}/(L_j\mathbb{Z}), \quad (3.1)$$

where $L_j = j!/\omega$ as in (1.12). Given a limit periodic function $f \in LP_\omega(\mathbb{R})$, let $f_j \in H^1(\mathbb{T}_j)$ be the periodization of f with period L_j as in (1.10). We assume that f and f_j satisfy the hypothesis in Theorem 1.9, in particular, (1.14). In view of the H^1 -global well-posedness of the defocusing NLS in the periodic setting, it follows that, for each $j \in \mathbb{N}$, there exists a unique global solution $u_j \in C(\mathbb{R}; H^1(\mathbb{T}_j))$ to (1.1) with $u_j|_{t=0} = f_j \in H^1(\mathbb{T}_j)$.

In the following, we show that $\{u_j\}_{j=1}^\infty$ is a Cauchy sequence in $C(\mathbb{R}; L^\infty(\mathbb{R}))$ endowed with the compact-open topology (in t with values in $L^\infty(\mathbb{R})$).⁴ For this purpose, we perform two kinds of scalings to f_j and

⁴ Recall that a sequence $\{w_j\}_{j=1}^\infty \subset C(\mathbb{R}_t; L^\infty(\mathbb{R}_x))$ converges to w in the compact-open topology if and only if, for every compact subset K of \mathbb{R}_t , the sequence $\{w_j(t)\}_{j=1}^\infty$ converges to $w(t)$ in $L^\infty(\mathbb{R}_x)$ uniformly in $t \in K$.

u_j . For $j \in \mathbb{N}$, set

$$g_j := \mathfrak{S}_{L_j}^{-1}[f_j] \quad \text{and} \quad v_j := \mathfrak{S}_{L_j}^{-1}[u_j],$$

where \mathfrak{S}_λ is the dilation operator defined in (2.5) and (2.6). We also set⁵

$$g^j := \mathfrak{S}_{L_{j+1}}^{-1}[f_j] \quad \text{and} \quad v^j := \mathfrak{S}_{L_{j+1}}^{-1}[u_j].$$

Note that v_j and v^j are global solutions to (1.1) on the standard torus \mathbb{T} with initial data g_j and g^j , respectively.

We first establish an estimate on $v^j - v_{j+1}$. By (2.9) and (1.14), we have

$$\|g^j\|_{H^1(\mathbb{T})} \lesssim L_{j+1}^{\frac{1}{2}+\frac{1}{k}} \|f_j\|_{H^1(\mathbb{T}_{j+1})} \leq C(\|f\|_{S^{1,2}(\mathbb{R})}) L_{j+1}^{1+\frac{1}{k}}, \quad (3.2)$$

$$\|g_{j+1}\|_{H^1(\mathbb{T})} \lesssim L_{j+1}^{\frac{1}{2}+\frac{1}{k}} \|f_{j+1}\|_{H^1(\mathbb{T}_{j+1})} \leq C(\|f\|_{S^{1,2}(\mathbb{R})}) L_{j+1}^{1+\frac{1}{k}}, \quad (3.3)$$

for sufficiently large $j \gg 1$. We also have

$$\begin{aligned} \|g^j - g_{j+1}\|_{H^1(\mathbb{T})} &\lesssim L_{j+1}^{\frac{1}{2}+\frac{1}{k}} \|f_j - f_{j+1}\|_{H^1(\mathbb{T}_{j+1})} \\ &\leq L_{j+1}^{\frac{1}{2}+\frac{1}{k}} (\|f_j - f\|_{H^1(\mathbb{T}_{j+1})} + \|f_{j+1} - f\|_{H^1(\mathbb{T}_{j+1})}) \\ &\lesssim e^{-L_{j+1}^{2k+\varepsilon}} L_{j+1}^{1+\frac{1}{k}}, \end{aligned} \quad (3.4)$$

for sufficiently large $j \gg 1$.

Let $2 \leq p \leq \infty$. Then, given a compact interval $I \subset \mathbb{R}$ with $|I| \geq 1$, we have the following Gagliardo-Nirenberg inequality

$$\|\phi\|_{L^p(I)} \lesssim \|\phi\|_{L^2(I)}^{\frac{1}{2}+\frac{1}{p}} \|\phi\|_{H^1(I)}^{\frac{1}{2}-\frac{1}{p}}. \quad (3.5)$$

See [6] for example. By a simple scaling argument, we can choose the implicit constant in (3.5) to be independent of I with $|I| \geq 1$. By (2.6), (3.5), and (1.14), we have

$$\begin{aligned} \|g^j\|_{L^{2k+2}(\mathbb{T})}^{2k+2} &= L_{j+1}^{1+\frac{2}{k}} \|f_j\|_{L^{2k+2}(\mathbb{T}_{j+1})}^{2k+2} \leq L_{j+1}^{2+\frac{2}{k}} \sup_{\substack{I \subset \mathbb{T}_{j+1} \\ |I|=1}} \|f_j\|_{L^{2k+2}(I)}^{2k+2} \\ &\lesssim L_{j+1}^{2+\frac{2}{k}} \sup_{\substack{I \subset \mathbb{T}_{j+1} \\ |I|=1}} \|f_j\|_{L^2(I)}^{k+2} \|f_j\|_{H^1(I)}^k \leq C(\|f\|_{S^{1,2}(\mathbb{R})}) L_{j+1}^{2+\frac{2}{k}} \end{aligned} \quad (3.6)$$

for sufficiently large $j \gg 1$. Hence, by the conservation of the Hamiltonian and the mass with (3.2) and (3.6), we obtain

$$\|v^j(t)\|_{H^1(\mathbb{T})} \leq C(\|f\|_{S^{1,2}(\mathbb{R})}) L_{j+1}^{1+\frac{1}{k}} \quad (3.7)$$

⁵Since f_j and u_j are periodic (in x) with period L_j , they are also periodic with period $L_{j+1} = (j+1)L_j$.

for any $t \in \mathbb{R}$ and sufficiently large $j \gg 1$. By a similar computation with (3.3), we also obtain

$$\|v_{j+1}(t)\|_{H^1(\mathbb{T})} \leq C(\|f\|_{S^{1,2}(\mathbb{R})})L_{j+1}^{1+\frac{1}{k}} \quad (3.8)$$

for any $t \in \mathbb{R}$ and sufficiently large $j \gg 1$.

Now, consider the Duhamel formulation of (1.1) on \mathbb{T} :

$$v^j(t) = \Gamma_{g^j} v^j(t) := e^{it\partial_x^2} g^j - i \int_0^t e^{i(t-t')\partial_x^2} |v^j|^{2k} v^j(t') dt'. \quad (3.9)$$

By the unitarity of the linear propagator and the algebra property of $H^1(\mathbb{T})$, one can easily show that the map Γ_{g^j} is a contraction on the ball of radius $2\|g^j\|_{H^1(\mathbb{T})}$ in $C([-T_*, T_*]; H^1(\mathbb{T}))$ for some $T_* > 0$. It follows from (3.2) that this standard fixed point argument via the Duhamel formulation (3.9) yields the local time T_* of existence, satisfying

$$T_* \sim \|g^j\|_{H^1(\mathbb{T})}^{-2k} \gtrsim L_{j+1}^{-2(k+1)}$$

for sufficiently large $j \gg 1$. By repeating this argument with (3.3), we see that the same argument holds even if we replace g^j and v^j with g_{j+1} and v_{j+1} , respectively. Note that, in view of the global-in-time control (3.7) and (3.8) of the H^1 -norms of the global solutions v^j and v_{j+1} , we can iterate this local-in-time argument indefinitely for both v^j and v_{j+1} on time intervals of size

$$T_* \sim L_{j+1}^{-2(k+1)}. \quad (3.10)$$

Finally, consider the difference of the Duhamel formulations for v^j and v_{j+1} . Then, by iterating the local argument on intervals of size T^* given by (3.10) and noting that the distance (in $H^1(\mathbb{T})$) between v^j and v_{j+1} can grow at most by a fixed constant multiple on each of $O(\frac{T}{T^*})$ many such intervals, there exists $J_* \gg 1$ and $c > 0$ such that

$$\|v^j - v_{j+1}\|_{C([-T, T]; H^1(\mathbb{T}))} \lesssim e^{[cTL_{j+1}^{2(k+1)}]+1} \|g^j - g_{j+1}\|_{H^1(\mathbb{T})} \quad (3.11)$$

for all $T > 0$ and $j \geq J_*$. Here, $[\tau]$ denotes the integer part of τ . Then, by undoing the scaling with (2.11), (3.11), and (3.4), we obtain

$$\begin{aligned} \|u_j - u_{j+1}\|_{C([-T, T]; L^\infty(\mathbb{R}))} &= \|u_j - u_{j+1}\|_{C([-T, T]; L^\infty(\mathbb{T}_{j+1}))} \\ &\lesssim L_{j+1}^{-\frac{1}{k}} \|v^j - v_{j+1}\|_{C([-L_{j+1}^{-2}T, L_{j+1}^{-2}T]; H^1(\mathbb{T}))} \\ &\lesssim e^{[cTL_{j+1}^{2k}]+1-L_{j+1}^{2k+\varepsilon}} L_{j+1} \longrightarrow 0, \end{aligned} \quad (3.12)$$

as $j \rightarrow \infty$, for each fixed $T > 0$. Hence, from (3.12) and (1.12), we have

$$\|u_j - u_{j'}\|_{C([-T, T]; L^\infty(\mathbb{R}))} \leq \sum_{\ell=j'}^{j-1} e^{-L_{\ell+1}^{2k}} \sim e^{-L_{j'+1}^{2k}}$$

for $j \geq j' \gg 1$, where the right-hand side converges to 0 as $j' \rightarrow \infty$. Therefore, $\{u_j\}_{j=1}^\infty$ converges in $C(\mathbb{R}; L^\infty(\mathbb{R}))$ with the compact-open topology, i.e. for each $T > 0$, the convergence is uniform in $|t| \leq T$. Denote the limit by $u \in C(\mathbb{R}; L^\infty(\mathbb{R}))$. Then, we have $u|_{t=0} = f$. Moreover, the above convergence implies that u is a distributional solution to (1.1), i.e. we have

$$\iint_{\mathbb{R} \times \mathbb{R}} u \left(-i\partial_t \phi + \partial_x^2 \phi \right) dx dt = \iint_{\mathbb{R} \times \mathbb{R}} |u|^{2k} u \cdot \phi dx dt$$

for any test function $\phi \in C_c^\infty(\mathbb{R} \times \mathbb{R})$.

Since $u_j(t) \in H^1(\mathbb{T}_j)$ for each $t \in \mathbb{R}$, it follows that $u_j(t)$ is continuous (in x) for each $t \in \mathbb{R}$ and $j \in \mathbb{N}$. Therefore, as a uniform limit of continuous periodic function $u_j(t)$, we conclude that $u(t)$ is limit periodic for each $t \in \mathbb{R}$.

We also claim that $u(t) \in LP_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$. Fix $t \in \mathbb{R}$. Given $\varepsilon > 0$, there exists $j \in \mathbb{N}$ such that

$$\|u(t) - u_j(t)\|_{L^\infty(\mathbb{R})} < \frac{\varepsilon}{3}. \quad (3.13)$$

By defining $u^{(L)}(t)$ and $u_j^{(L)}(t)$ be the periodizations of $u(t)$ and $u_j(t)$ as in (1.21), it follows from Lemma 1.15 that there exists $N \in \mathbb{N}$ such that

$$\sup_{x \in \mathbb{R}} |u_*^{(L)}(t, x) - A_{n,L}[u_*](t, x)| < \frac{\varepsilon}{3}, \quad (3.14)$$

for all $n \geq N$, where $u_* = u$ or u_j . Here, $A_{n,L}$ is the averaging operator defined in (1.20). Then, from (3.13) and (3.14) with (1.20), we see that

$$\|u^{(L)}(t) - u_j^{(L)}(t)\|_{L^\infty(\mathbb{R})} < \varepsilon. \quad (3.15)$$

Now, let $L \in \mathbb{R} \setminus (\mathbb{Q}/\omega)$. Then, noting that $u_j(t)$ is periodic (in x) with period $L_j = j!/\omega \in \mathbb{Q}/\omega$, it follows from Lemma 1.15 that

$$u_j^{(L)}(t) \equiv 0. \quad (3.16)$$

Hence, from (3.15) and (3.16), we have

$$\|u^{(L)}(t)\|_{L^\infty(\mathbb{R})} < \varepsilon.$$

Since the choice of $\varepsilon > 0$ was arbitrary, we conclude that

$$u^{(L)}(t) \equiv 0.$$

In particular, it follows from Lemma 1.15 that $1/L \notin \sigma(u(t))$. Therefore, we have $\sigma(u(t)) \subset \omega \cdot \mathbb{Q}$ for any $t \in \mathbb{R}$. This proves that $u(t) \in LP_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$.

Next, we prove uniqueness of the solution u constructed above. Given $\omega \in \mathbb{R} \setminus \{0\}$, let $f \in LP_\omega(\mathbb{R})$ satisfy the hypothesis of Theorem 1.9. Then, it follows from Remark 1.13 that f is given by its Fourier series (1.9). Set

$$\boldsymbol{\omega} := \{r_m \omega\}_{m=1}^\infty \in \mathbb{R}^{\mathbb{N}},$$

where r_m is as in (1.9). Then, uniqueness follows from the local well-posedness result in $\mathcal{A}_\omega(\mathbb{R})$ presented in [16], once we show that $u(t) \in \mathcal{A}_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$.

• **Case (a):** $t = 0$.

Let $A(j)$ be as in (1.11). Then, we have

$$\mathbb{N} = \bigcup_{j \in \mathbb{N}} A(j) = A(1) \cup \bigcup_{j \in \mathbb{N}} \{A(j+1) \setminus A(j)\} \quad (3.17)$$

since $r_m \in \mathbb{Q}$ satisfies (i) $r_m j! \in \mathbb{Z}$ for some $j = j(r_m) \in \mathbb{N}$ and (ii) $r_m \tilde{j}! \in \mathbb{Z}$ for all $\tilde{j} \geq j(r_m)$.

It follows from (2.12) and (1.14) that there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} \sum_{m \in A(j+1) \setminus A(j)} |\widehat{f}(r_m \omega)| &\lesssim \|f_{j+1} - f_j\|_{H^1(\mathbb{T}_{j+1})} \leq L_{j+1}^{\frac{1}{2}} \|f_{j+1} - f_j\|_{S^{1,2}(\mathbb{R})} \\ &\leq L_{j+1}^{\frac{1}{2}} (\|f_{j+1} - f\|_{S^{1,2}(\mathbb{R})} + \|f - f_j\|_{S^{1,2}(\mathbb{R})}) \\ &\lesssim e^{-L_{j+1}^{2k}} \end{aligned} \quad (3.18)$$

for all $j \geq j_0$. Similarly, we have

$$\begin{aligned} \|f_{j_0}\|_{\mathcal{A}_\omega(\mathbb{R})} &\lesssim \|f_{j_0}\|_{H^1(\mathbb{T}_{j_0})} \leq L_{j_0}^{\frac{1}{2}} (\|f\|_{S^{1,2}(\mathbb{R})} + \|f_{j_0} - f\|_{S^{1,2}(\mathbb{R})}) \\ &\lesssim L_{j_0}^{\frac{1}{2}} \|f\|_{S^{1,2}(\mathbb{R})} \end{aligned} \quad (3.19)$$

for sufficiently large $j_0 \gg 1$. Then, from (3.18) and (3.19) with (1.12), we obtain

$$\begin{aligned} \|f\|_{\mathcal{A}_\omega(\mathbb{R})} &= \sum_{m=1}^{\infty} |\widehat{f}(r_m \omega)| = \sum_{m \in A_{j_0}} |\widehat{f}(r_m \omega)| + \sum_{j=j_0}^{\infty} \sum_{m \in A(j+1) \setminus A(j)} |\widehat{f}(r_m \omega)| \\ &\lesssim \|f_{j_0}\|_{\mathcal{A}_\omega(\mathbb{R})} + \sum_{j=j_0}^{\infty} e^{-L_{j+1}^{2k}} \lesssim L_{j_0}^{\frac{1}{2}} \|f\|_{S^{1,2}(\mathbb{R})} + e^{-L_{j_0+1}^{2k}} < \infty. \end{aligned}$$

Therefore, we conclude that $f \in \mathcal{A}_\omega(\mathbb{R})$.

• **Case (b):** $t \neq 0$.

Fix $t \in \mathbb{R} \setminus \{0\}$. Then, by slightly modifying the computations in (3.12), we have

$$\begin{aligned} \|u_j - u_{j+1}\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} &\leq \|u_j - u_{j+1}\|_{C([-T, T]; H^1(\mathbb{T}_{j+1}))} \\ &\lesssim L_{j+1}^{\frac{1}{2} - \frac{1}{k}} \|v^j - v_{j+1}\|_{C([-L_{j+1}^{-2} T, L_{j+1}^{-2} T]; H^1(\mathbb{T}))} \\ &\lesssim e^{[cTL_{j+1}^{2k}] + 1 - L_{j+1}^{2k+\epsilon}} L_{j+1}^{\frac{3}{2}} \longrightarrow 0, \end{aligned} \quad (3.20)$$

as $j \rightarrow \infty$, for each $T > 0$. In particular, $u_j(t)$ converges to $u(t)$ in $S^{1,2}(\mathbb{R})$, uniformly on the time interval $[-T, T]$ for each $T > 0$. Therefore, we conclude that $u \in C(\mathbb{R}; S^{1,2}(\mathbb{R}))$.

Fix $j \in \mathbb{N}$. Let A_{n,L_j} be the averaging operator defined in (1.20) and $u^{(L_j)}(t)$ and $u_j^{(L_j)}(t)$ be the periodizations of $u(t)$ and $u_j(t)$ with period L_j defined in (1.21). By Lemma 1.15, $A_{n,L_j}[u(t) - u_j(t)]$ converges uniformly (in x) to $u^{(L_j)}(t) - u_j^{(L_j)}(t)$ as $n \rightarrow \infty$. In particular, given an interval $I \subset \mathbb{R}$ with $|I| = 1$, $A_{n,L_j}[u(t) - u_j(t)]$ converges $u^{(L_j)}(t) - u_j^{(L_j)}(t)$ in $L^2(I)$. Moreover, we have

$$\|A_{n,L_j}[u(t) - u_j(t)]\|_{H^1(I)} \leq \|u(t) - u_j(t)\|_{S^{1,2}(\mathbb{R})}.$$

Namely, $\{A_{n,L_j}[u(t) - u_j(t)]\}_{n \in \mathbb{N}}$ is bounded in $H^1(I)$. Therefore, $A_{n,L_j}[u(t) - u_j(t)]$ converges weakly in $H^1(I)$ as $n \rightarrow \infty$.⁶ As a result, we obtain

$$\begin{aligned} \|u^{(L_j)}(t) - u_j^{(L_j)}(t)\|_{S^{1,2}(\mathbb{R})} &= \left\| \lim_{n \rightarrow \infty} A_{n,L_j}[u(t) - u_j(t)] \right\|_{S^{1,2}(\mathbb{R})} \\ &\leq \liminf_{n \rightarrow \infty} \|A_{n,L_j}[u(t) - u_j(t)]\|_{S^{1,2}(\mathbb{R})} \\ &\leq \|u(t) - u_j(t)\|_{S^{1,2}(\mathbb{R})}. \end{aligned} \quad (3.21)$$

Since $u_j(t)$ is already periodic with period L_j , we have $u_j^{(L_j)}(t) = u_j(t)$. Then, it follows from the triangle inequality with (3.21) and (3.20) that there exists $j_0 \in \mathbb{N}$ such that

$$\begin{aligned} &\|u^{(L_{j+1})}(t) - u^{(L_j)}(t)\|_{H^1(\mathbb{T}_{j+1})} \\ &\leq (\|u^{(L_{j+1})}(t) - u_{j+1}^{(L_{j+1})}(t)\|_{H^1(\mathbb{T}_{j+1})} \\ &\quad + \|u_{j+1}(t) - u_j(t)\|_{H^1(\mathbb{T}_{j+1})} + \|u_j^{(L_j)}(t) - u^{(L_j)}(t)\|_{H^1(\mathbb{T}_{j+1})}) \\ &\leq L_{j+1}^{\frac{1}{2}} (\|u^{(L_{j+1})}(t) - u_{j+1}^{(L_{j+1})}(t)\|_{S^{1,2}(\mathbb{R})} \\ &\quad + \|u_{j+1}(t) - u_j(t)\|_{S^{1,2}(\mathbb{R})}) + \|u_j^{(L_j)}(t) - u^{(L_j)}(t)\|_{S^{1,2}(\mathbb{R})} \\ &\leq L_{j+1}^{\frac{1}{2}} (\|u(t) - u_{j+1}(t)\|_{S^{1,2}(\mathbb{R})} \\ &\quad + \|u_{j+1}(t) - u_j(t)\|_{S^{1,2}(\mathbb{R})}) + \|u_j(t) - u(t)\|_{S^{1,2}(\mathbb{R})} \\ &\lesssim e^{-L_{j+1}^{2k}} \end{aligned} \quad (3.22)$$

⁶Suppose that $\{f_n\}_{n=1}^\infty$ converges to f in L^2 and is bounded in H^1 . Fix $\varepsilon > 0$. Given a test function $\phi \in H^{-1}$, let $\phi_\varepsilon \in L^2$ such that $\|\phi - \phi_\varepsilon\|_{H^{-1}} < \varepsilon/(2M)$, where $M = \sup_n \|f_n - f\|_{H^1}$. Then, we have

$$|\langle f_n - f, \phi \rangle| \leq \|f_n - f\|_{L^2} \|\phi_\varepsilon\|_{L^2} + \frac{1}{2}\varepsilon < \varepsilon$$

for $n \geq N = N(\varepsilon, \phi)$.

for all $j \geq j_0$. Finally, proceeding as in (3.19) with (3.22), we obtain

$$\begin{aligned}
\|u(t)\|_{\mathcal{A}_\omega(\mathbb{R})} &= \sum_{m=1}^{\infty} |\widehat{u}(t, r_m \omega)| \\
&= \sum_{m \in A_{j_0}} |\widehat{u}(t, r_m \omega)| + \sum_{j=j_0}^{\infty} \sum_{m \in A(j+1) \setminus A(j)} |\widehat{u}(t, r_m \omega)| \\
&\lesssim \|u^{(L_{j_0})}(t)\|_{H^1(\mathbb{T}_{j_0})} + \sum_{j=j_0}^{\infty} \|u^{(L_{j+1})}(t) - u^{(L_j)}(t)\|_{H^1(\mathbb{T}_{j+1})} \\
&\leq L_{j_0}^{\frac{1}{2}} \|u(t)\|_{S^{1,2}(\mathbb{R})} + C e^{-cL_{j_0}^{2k}+1} < \infty.
\end{aligned}$$

Therefore, we conclude that $u(t) \in \mathcal{A}_\omega(\mathbb{R})$ for each $t \in \mathbb{R}$.

Lastly, we present the proof of Theorem 1.9 (ii). Given $J \in \mathbb{N}$ and $K > 0$, let $\mathcal{B}^\omega(J, K)$ be as in (1.15). Fix $T > 0$ and $\varepsilon > 0$. Given $f, \tilde{f} \in \mathcal{B}^\omega(J, K)$, let u and \tilde{u} be the global solutions to (1.1) constructed above with f and \tilde{f} as initial data, respectively. Denoting the periodizations of f and \tilde{f} with period L_j by f_j and \tilde{f}_j , we denote by u_j and \tilde{u}_j the global solutions to (1.1) on \mathbb{T}_j with initial data f_j and \tilde{f}_j , respectively. Then, it follows from (3.20) that there exists $J_1 \in \mathbb{N}$ such that

$$\|u - u_j\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} + \|\tilde{u} - \tilde{u}_j\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} < \frac{1}{2}\varepsilon \quad (3.23)$$

for all $j \geq J_1$.

Let $v_j = \mathfrak{S}_{L_j}^{-1}[u_j]$ and $\tilde{v}_j = \mathfrak{S}_{L_j}^{-1}[\tilde{u}_j]$. By proceeding as in (3.20), we have

$$\begin{aligned}
\|u_j - \tilde{u}_j\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} &\leq \|u_j - \tilde{u}_j\|_{C([-T, T]; H^1(\mathbb{T}_j))} \\
&\lesssim L_j^{\frac{1}{2} - \frac{1}{k}} \|v_j - \tilde{v}_j\|_{C([-L_j^{-2}T, L_j^{-2}T]; H^1(\mathbb{T}))}. \quad (3.24)
\end{aligned}$$

Note that v_j and \tilde{v}_j satisfy the global H^1 -bound (3.8) (with $j+1$ replaced by j), where the constant depends only on K . Hence, by iterating the local-in-time argument over time intervals of size $T_* \sim L_j^{-2(k+1)}$, we obtain

$$\begin{aligned}
\|v_j - \tilde{v}_j\|_{C([-L_j^{-2}T, L_j^{-2}T]; H^1(\mathbb{T}))} &\leq e^{[cTL_j^{2k}] + 1} \|v_j(0) - \tilde{v}_j(0)\|_{H^1(\mathbb{T})} \\
&\leq e^{[cTL_j^{2k}] + 1} L_j^{\frac{1}{2} + \frac{1}{k}} \|f_j - \tilde{f}_j\|_{H^1(\mathbb{T}_j)} \leq e^{[cTL_j^{2k}] + 1} L_j^{1 + \frac{1}{k}} \|f_j - \tilde{f}_j\|_{S^{1,2}(\mathbb{R})} \\
&\leq e^{[cTL_j^{2k}] + 1} L_j^{1 + \frac{1}{k}} (\|f_j - f\|_{S^{1,2}(\mathbb{R})} \\
&\quad + \|f - \tilde{f}\|_{S^{1,2}(\mathbb{R})} + \|\tilde{f} - \tilde{f}_j\|_{S^{1,2}(\mathbb{R})}). \quad (3.25)
\end{aligned}$$

It follows from (3.24) and (3.25) with (1.14) that there exists $J_2 \in \mathbb{N}$ such that

$$\|u_j - \tilde{u}_j\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} \leq e^{[cTL_j^{2k}] + 1} L_j^{\frac{3}{2}} \|f - \tilde{f}\|_{S^{1,2}(\mathbb{R})} + \frac{1}{4}\varepsilon \quad (3.26)$$

for all $j \geq J_2$.

Finally, letting $j_* = \max(J_1, J_2)$, it follows from (3.23) and (3.26) that

$$\|u - \tilde{u}\|_{C([-T, T]; S^{1,2}(\mathbb{R}))} \leq e^{[cTL_{j_*}^{2k}] + 1} L_{j_*}^{\frac{3}{2}} \|f - \tilde{f}\|_{S^{1,2}(\mathbb{R})} + \frac{3}{4}\varepsilon < \varepsilon,$$

where the last inequality holds as long as we have

$$\|f - \tilde{f}\|_{S^{1,2}(\mathbb{R})} < \delta = \delta(\varepsilon, j_*) \ll 1.$$

Note that by choosing $J_1, J_2 \geq J$, we can make sure that they do not depend on a particular choice of functions in $\mathcal{B}^\omega(J, K)$. This proves continuous dependence of the flow on $\mathcal{B}^\omega(J, K)$.

Remark 3.1. In the following, we briefly discuss how Theorem 1.9 also follows in the focusing case if $k = 1$. Let $\phi \in H^1(\mathbb{T})$. Then, it follows from (3.5) that there exists $C_0 > 0$ such that

$$\|\phi\|_{H^1(\mathbb{T})}^2 \lesssim M[\phi] + H[\phi] + C_0(M[\phi])^3$$

In particular, by the conservation of the Hamiltonian and the mass along with (2.8), we have

$$\begin{aligned} \|v^j(t)\|_{H^1(\mathbb{T})} &\leq \left(M[g^j] + H[g^j] + C_0(M[g^j])^3 \right)^{\frac{1}{2}} \\ &\leq C(\|f\|_{S^{1,2}(\mathbb{R})}) L_{j+1}^3 \end{aligned} \tag{3.27}$$

in place of (3.7). A similar computation shows that (3.8) also holds with L_{j+1}^3 . As a result, we obtain $T^* \sim L_{j+1}^{-6}$ instead of (3.10). Then, it is easy to see that the rest of the proof of Theorem 1.9 goes through with small modifications, as long as (1.16) holds.

APPENDIX A. ON THE CONDITION (1.14)

In this appendix, we briefly investigate a meaning of the condition (1.14) on the rate of convergence of f_j to a limit periodic initial condition f . Given a quasi-periodic function f of the form (1.6), one can talk about a decay of the Fourier coefficients of the form: $|\hat{f}(\omega \cdot \mathbf{n})| \lesssim |\mathbf{n}|^{-\gamma}$ and $|\hat{f}(\omega \cdot \mathbf{n})| \lesssim \exp(-\kappa|\mathbf{n}|^\theta)$ for $\mathbf{n} \in \mathbb{Z}^N$. For a generic almost periodic function, however, such conditions do not make sense since $|\mathbf{n}| = \infty$ for $\mathbf{n} \in \mathbb{Z}^N$, unless $\mathbf{n} = (n_1, n_2, \dots)$ has a finite support. Since our limit periodic initial condition f in Theorem 1.9 is not quasi-periodic, it does not seem appropriate or at least seems non-trivial to characterize the condition (1.14) in terms of a decay of the Fourier coefficients only in $|\mathbf{n}|$. We instead consider a sufficient condition for (1.14) and discuss a decay of the Fourier coefficients in n_j (see (A.1) below for the definition of n_j) and in L_j (and hence in j) in the following.

Let $A(j)$ be as in (1.11) with the understanding that $A(0) = \emptyset$. With (1.9) and Remark 1.13, we have

$$f(x) = \sum_{m=1}^{\infty} \widehat{f}(r_m \omega) e^{2\pi i r_m \omega x} = \sum_{j=1}^{\infty} \sum_{m \in A(j) \setminus A(j-1)} \widehat{f}(r_m \omega) e^{2\pi i r_m \omega x}.$$

Here, each summand in the j -summation is given by $f_j - f_{j-1}$ defined in (1.10), and thus is periodic with period $L_j = j!/\omega$. With $n_j = r_m j!$, we have

$$f(x) = \sum_{j=1}^{\infty} \sum_{\substack{n_j \in \mathbb{Z} \\ j \nmid n_j}} \widehat{f}\left(\frac{n_j}{L_j}\right) e^{2\pi i \frac{n_j}{L_j} x}. \quad (\text{A.1})$$

Note that we have

$$\widehat{f}\left(\frac{n_j}{L_j}\right) = \widehat{f}_j\left(\frac{n_j}{L_j}\right),$$

where the Fourier transform on the right-hand side is that for periodic functions with period L_j discussed in Section 2. With $\mathbf{n} = (n_1, n_2, \dots)$ and $\boldsymbol{\omega} = (L_1^{-1}, L_2^{-1}, \dots)$, we see that (A.1) is an analogous formulation to (1.6) in our limit periodic setting, showing that we have infinite many summations over \mathbb{Z} unlike the quasi-periodic setting.

In the remaining part, we consider the following sufficient condition for (1.14):

$$\|f_j - f_{j-1}\|_{S^{1,2}(\mathbb{R})} \leq C e^{-L_j^{2k+\varepsilon}} \quad (\text{A.2})$$

for all sufficiently large j . This in turn is guaranteed if we have

$$\|f_j - f_{j-1}\|_{H^1(\mathbb{T}_j)} \leq C e^{-L_j^{2k+\varepsilon}}, \quad (\text{A.3})$$

where \mathbb{T}_j as in (3.1). Hence, by letting $F_j = f_j - f_{j-1}$ denote the difference of consecutive periodizations f_j and f_{j-1} (with the understanding that $f_0 \equiv 0$), we see that the condition (1.14) is satisfied if (i) $f = \sum_{j=1}^{\infty} F_j$ and (ii) the $H^1(\mathbb{T}_j)$ -norms of the L_j -periodic functions F_j decay at a rate $C e^{-L_j^{2k+\varepsilon}}$ for all sufficiently large j .

Lastly, note that the condition (A.3) is essentially necessary for (A.2). Indeed, it follows from (A.1) and (A.2) that

$$C e^{-L_j^{2k+\varepsilon}} \geq \|f_j - f_{j-1}\|_{S^{1,2}(\mathbb{R})} \geq L_j^{-\frac{1}{2}} \|f_j - f_{j-1}\|_{H^1(\mathbb{T}_j)}.$$

Hence, from (2.4), we must have

$$\left(\sum_{\substack{n_j \in \mathbb{Z} \\ j \nmid n_j}} \left\langle \frac{n_j}{L_j} \right\rangle^2 |\widehat{f}\left(\frac{n_j}{L_j}\right)|^2 \right)^{\frac{1}{2}} \leq C' e^{-L_j^{2k+\frac{\varepsilon}{2}}} \quad (\text{A.4})$$

for all sufficiently large j , if (A.2) holds. The condition (A.4) states that the Fourier coefficients $\widehat{f}(\frac{n_j}{L_j})$ must decay polynomially in n_j (in an average sense) besides the very fast decay in j for each L_j -periodic component.

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REFERENCES

- [1] A. Besicovitch, *Almost periodic functions*, Dover Publications, Inc., New York, 1955. xiii+180 pp.
- [2] H. Bohr, *Zur theorie der fast periodischen funktionen. I. Eine verallgemeinerung der theorie der fourierreihen*, Acta Math. 45 (1925), no. 1, 29–127.
- [3] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations*, Geom. Funct. Anal. 3 (1993), no. 2, 107–156.
- [4] J. Bourgain, *A remark on normal forms and the "I-method" for periodic NLS*, J. Anal. Math. 94 (2004), 125–157.
- [5] A. Boutet de Monvel, I. Egorova, *On solutions of nonlinear Schrödinger equations with Cantor-type spectrum*, J. Anal. Math. 72 (1997), 1–20.
- [6] H. Brezis, *Functional analysis*, Sobolev spaces and partial differential equations. Universitext. Springer, New York, 2011. xiv+599 pp.
- [7] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. xiv+323 pp.
- [8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global Well-Posedness for KdV and Modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc. 16 (2003), no. 3, 705–749.
- [9] C. Corduneanu, *Almost periodic functions*, With the collaboration of N. Gheorghiu and V. Barbu. Translated from the Romanian by Gitta Bernstein and Eugene Tomer. Interscience Tracts in Pure and Applied Mathematics, No. 22. Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968. x+237 pp.
- [10] D. Damanik, M. Goldstein, *On the existence and uniqueness of global solutions for the KdV equation with quasi-periodic initial data*, arXiv:1212.2674v2 [math.AP].
- [11] I. Egorova, *The Cauchy problem for the KdV equation with almost periodic initial data whose spectrum is nowhere dense*, Spectral operator theory and related topics, 181–208, Adv. Soviet Math., 19, Amer. Math. Soc., Providence, RI, 1994.
- [12] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations. III. Special theories in dimensions 1, 2 and 3*, Ann. Inst. H. Poincaré Sect. A (N.S.) 28 (1978), no. 3, 287–316.
- [13] Y. Katznelson, *An introduction to harmonic analysis*, Third edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2004. xviii+314 pp.
- [14] T. Ogawa, Y. Tsutsumi, *Blow-up of solutions for the nonlinear Schrödinger equation with quartic potential and periodic boundary condition*, Functional-analytic methods for partial differential equations (Tokyo, 1989), 236–251, Lecture Notes in Math., 1450, Springer, Berlin, 1990.
- [15] T. Ogawa, Y. Tsutsumi, *Blow-up of H^1 solutions for the one-dimensional nonlinear Schrödinger equation with critical power nonlinearity*, Proc. Amer. Math. Soc. 111 (1991), no. 2, 487–496.

- [16] T. Oh, *On nonlinear Schrödinger equations with almost periodic initial data*, to appear in SIAM J. Math. Anal.
- [17] T. Tao, *Nonlinear dispersive equations. Local and global analysis*, CBMS Regional Conference Series in Mathematics, 106. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. xvi+373 pp.
- [18] K. Tsugawa, *Local well-posedness of the KdV equation with quasi-periodic initial data*, SIAM J. Math. Anal. 44 (2012), no. 5, 3412–3428.
- [19] Y. Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkcial. Ekvac. 30 (1987), no. 1, 115–125.

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